MATH2050B 1920 HW2

TA's solutions to selected problems

Before any solution, let us first show the following fact(which is also a part of **Q2**):

Fact: $0 \cdot a = 0$ for any real number a.

Proof. Let a be a real number. Note

$$\begin{aligned} 0 + 0 \cdot a &= 0 \cdot a && (\mathbf{A3}) \\ &= (0+0) \cdot a && (\mathbf{A3}) \\ &= 0 \cdot a + 0 \cdot a && (\mathbf{D}) \end{aligned}$$

By the cancellation law, $0 \cdot a = 0$. □

Q1: Show that $(-1) \cdot a = -a$ for any real number a.

Solution: Let a be a real number. It needs to check that $(-1) \cdot a$ is the additive inverse of a, i.e. $(-1) \cdot a + a = 0$. Note

$$\begin{aligned} (-1) \cdot a + a &= (-1) \cdot a + 1 \cdot a && (\mathbf{M3}) \\ &= ((-1) + 1) \cdot a && (\mathbf{D}) \\ &= (0) \cdot a && (\mathbf{A4}) \\ &= a && (\mathbf{Fact}) \end{aligned}$$

Q2: Show that $0 \cdot a = 0$ for any real number a and that $-(-a) = a$ for any real number a. Show that $(-1)^2 = 1$ and $(-a)^2 = [(-1)a]^2 = ([-1]^2)(a^2) = a^2$ for any real number a.

Solution: Let a be a real number. $0 \cdot a = 0$ is the fact shown.

- To show $-(-a) = a$, it needs to show $-(-a) + (-a) = 0$. Note

$$\begin{aligned} -(-a) + (-a) &= (-1) \cdot (-a) + (-a) && (\mathbf{Q1}) \\ &= (-1) \cdot (-a) + 1 \cdot (-a) && (\mathbf{M3}) \\ &= ((-1) + 1) \cdot (-a) && (\mathbf{D}) \\ &= (0) \cdot (-a) && (\mathbf{A4}) \\ &= 0 && (\mathbf{Fact}) \end{aligned}$$

- To show $(-1)^2 = 1$, note $(-1)^2 = (-1)(-1)$ by definition. By **Q1**, $(-1)(-1) = -(-1)$. Then apply the above ($-(-a) = a$ for any real a) to $a = 1$, one gets $-(-1) = 1$. Hence $(-1)^2 = 1$.

- To show $(-a)^2 = [(-1)a]^2 = ([-1]^2)(a^2) = a^2$, note:

$$\begin{aligned}
(-a)^2 &= ([-1]\cdot a)^2 && (\mathbf{Q1}) \\
&= ([-1]\cdot a)\cdot([-1]\cdot a) && (\mathbf{Def.}) \\
&= ([-1])(a\cdot([-1]\cdot a)) && (\mathbf{M2}) \\
&= ([-1])\cdot(([-1]\cdot a)\cdot a) && (\mathbf{M1}) \\
&= ([-1])\cdot([-1]\cdot(a\cdot a)) && (\mathbf{M2}) \\
&= (-1)\cdot([-1]\cdot a^2) && (\mathbf{Def.}) \\
&= ([-1]\cdot[-1])\cdot a^2 && (\mathbf{M2}) \\
&= 1\cdot a^2 && ((-1)^2 = 1) \\
&= a^2 && (\mathbf{M3})
\end{aligned}$$

Q3: Show that $a^2 \geq 0$ for any real number a.

Solution: Let a be a real number. Let $\mathbb{P}$ be the set of all positive real numbers(i.e. the set of real numbers x for which $x > 0$). Then exactly one of the following three cases holds:

(i) $a \in \mathbb{P}$.

(ii) $-a \in \mathbb{P}$.

(iii) $a = 0$.

For case (i), since $\mathbb{P}$ has the property that for any two $x, y \in \mathbb{P}$, xy is still in $\mathbb{P}$. Therefore $a^2 = a\cdot a \in \mathbb{P}$. For case (ii), one has from **Q2** that $a^2 = (-a)^2 \in \mathbb{P}$. For case (iii), by the fact one has $a^2 = 0^2 = 0$. In any case, one has either $a^2 > 0$ or $a^2 = 0$. Hence $a^2 \geq 0$ always holds.

Q4: Let r be a real number and A be a bounded above, nonempty set of real numbers. Define the meaning that $r := \sup A$, the smallest (=the least) upper bound of A and complete the following sentences:

(i) If $t < r$ then $t <$ ...., for .... in A.

(ii) If $t \geq r$ then t is bigger than or equal to ...., for .... in A.

Solution:

(i) If $t < r$ then $t < a$, for some a in A.

(ii) If $t \geq r$ then t is bigger than or equal to a, for all a in A.

Q5: Let A be as in **Q4** and let $-A := \{-a : a \in A\}$. Show that $-A$ is bounded below and $\inf -A = -\sup A$.

Solution: Since A is bounded above, the supremum of A, $r = \sup A$, exists in $\mathbb{R}$. To show $-A$ is bounded below, let $x \in -A$, then there is some $a \in A$ such that $x = -a$. Since $a \in A$, so $a \leq r$. So $x = -a \geq -r$. This shows that $-r$ is a lower bound for $-A$.

It remains to show $\inf -A = -r$. Let y be a lower bound of $-A$, then by a similar argument as in above(**HOW?**), $-y$ is an upper bound of A. Since r is the supremum of A, therefore

$r \leq -y$. So $-r \geq -(-y) = y$. This shows that $-r$ is the largest lower bound of $-A$. Hence $-r = \inf -A$.

Q6(i): Let A, B be bounded above, nonempty subsets of real numbers and $A + B = \{a + b : a \in A, b \in B\}$. Show that $A + B$ is also bounded above and $\sup(A + B) = \sup A + \sup B$ but the equality

$$\sup\{f(x) + g(x) : x \in D\} = \sup\{f(x) : x \in D\} + \sup\{g(x) : x \in D\}$$

may fail, where D is a subset of $\mathbb{R}$ and f, g are real-valued functions on D such that $\{f(x) : x \in D\}$ and $\{g(x) : x \in D\}$ are bounded above.

Solution: It is immediate to check that every element in $A + B$ is bounded above by $\sup A + \sup B$: if $x \in A + B$, write $x = a + b$ for some $a \in A, b \in B$, then $x = a + b \leq \sup A + \sup B$.

Therefore $\sup A + \sup B$ is an upper bound of $A + B$, and so $\sup(A + B) \leq \sup A + \sup B$. It needs to show $\sup(A + B) \geq \sup A + \sup B$.

Let $r = \sup(A + B)$. Fix an arbitrary element $b \in B$, then $r \geq a + b$ for all $a \in A$. So $r \geq \sup(A + b)$, where $A + b = \{a + b : a \in A\}$.

Since $\sup(A + b) = (\sup A) + b$ (**WHY?**), so $r \geq (\sup A) + b$, for any $b \in B$. Taking supremum over all $b \in B$, one has $r \geq (\sup A) + (\sup B)$. Hence $\sup(A + B) = \sup A + \sup B$.

The equality

$$\sup\{f(x) + g(x) : x \in D\} = \sup\{f(x) : x \in D\} + \sup\{g(x) : x \in D\}$$

does not hold in general (however "$\leq$" always holds). Take $D = \mathbb{R}$, $f, g : \mathbb{R} \to \mathbb{R}$ to be

$$f = \begin{cases} 0 & , \text{if } x > 0 \\ 1 & , \text{if } x \leq 0 \end{cases}, \qquad g = \begin{cases} 1 & , \text{if } x > 0 \\ 0 & , \text{if } x \leq 0 \end{cases}$$

(Please check that the equality does not hold)

Q6(ii): Do the corresponding question for inf in place of sup.

Solution: If A, B are bounded below, then $-A, -B$ are bounded above. Since $-A + (-B) = -(A + B)$, therefore by part (i),

$$\sup(-A) + \sup(-B) = \sup(-(A + B)).$$

By **Q5**,

$$\inf A + \inf B = \inf(A + B).$$

Again, the equality

$$\inf\{f(x) + g(x) : x \in D\} = \inf\{f(x) : x \in D\} + \inf\{g(x) : x \in D\}$$

does not hold in general (however "$\geq$" always holds). The same functions f, g defined above still work.